

NOMURA ALGEBRAS OF NONSYMMETRIC HADAMARD MODELS

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ABSTRACT. We show that the Nomura algebra of the nonsymmetric Hadamard model coincides with the Bose–Mesner algebra of the directed Hadamard graph.

1. INTRODUCTION

Spin models for link invariants were introduced by Jones [7], and their connection to combinatorics was revealed first in [4]. Jaeger and Nomura [6] constructed nonsymmetric spin models for link invariants from Hadamard matrices, and showed that these models give link invariants which depend nontrivially on link orientation. These models are a modification of the Hadamard model originally constructed by Nomura [9]. Jaeger and Nomura also pointed out a similarity between the association scheme of a Hadamard graph and the association scheme containing their new nonsymmetric spin model.

Nomura [10], and later Jaeger, Matsumoto and Nomura [5] introduced an algebra called the Nomura algebra of a type II matrix W , and showed that this algebra coincides with the Bose–Mesner algebra of some self-dual association schemes when W is a spin model. By [5] the Nomura algebra of the Hadamard model coincides with the Bose–Mesner algebra of the corresponding Hadamard graph.

The purpose of this paper is to determine the Nomura algebra of a nonsymmetric Hadamard model to be the Bose–Mesner algebra of the corresponding directed Hadamard graph. We also show that the directed Hadamard graph can be constructed from the ordinary Hadamard graphs by means of a general method given by Klin, Muzychuk, Pech, Woldar and Zieschang [8].

2. PRELIMINARIES ON NOMURA ALGEBRAS

Let X be a finite set with k elements. We denote by $\text{Mat}_X(\mathbf{C})$ the algebra of square matrices with complex entries whose rows and columns are indexed by X . We also denote by $\text{Mat}_X(\mathbf{C}^*)$ the subset

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of $\text{Mat}_X(\mathbf{C})$ consisting of matrices all of whose entries are nonzero. For $W \in \text{Mat}_X(\mathbf{C})$ and $x, y \in X$, the (x, y) -entry of W is denoted by $W(x, y)$.

A *type II matrix* is a matrix $W \in \text{Mat}_X(\mathbf{C}^*)$ which satisfies the *type II condition*:

$$(1) \quad \sum_{x \in X} \frac{W(a, x)}{W(b, x)} = k\delta_{a,b} \quad (\text{for all } a, b \in X).$$

For a type II matrix $W \in \text{Mat}_X(\mathbf{C}^*)$ and $a, b \in X$, we define a column vector $Y_{ab} \in \mathbf{C}^X$ whose x -entry is given by

$$Y_{ab}(x) = \frac{W(x, a)}{W(x, b)}.$$

The Nomura algebra $\mathcal{N}(W)$ of W is defined by

$$\mathcal{N}(W) = \{M \in \text{Mat}_n(\mathbf{C}^*) \mid Y_{ab} \text{ is an eigenvector for } M \text{ for all } a, b \in X\}.$$

A type II matrix W is called a *spin model* if it satisfies the *type III condition*:

$$\sum_{x \in X} \frac{W(a, x)W(b, x)}{W(c, x)} = d \frac{W(a, b)}{W(a, c)W(c, b)} \quad (\text{for all } a, b, c \in X),$$

where $d^2 = k$. It is shown in [5, Theorem 11] that if W is a spin model, then $\mathcal{N}(W)$ is the Bose-Mesner algebra of some self-dual association scheme.

We shall associate with W an undirected graph G on the vertex set $X \times X$. Given two column vectors T, T' indexed by X , we write $\langle T, T' \rangle$ for their Hermitian scalar product $\sum_{x \in X} T(x)\overline{T'(x)}$. Two distinct vertices $(a, b), (c, d)$ will be adjacent in G iff $\langle Y_{ab}, Y_{cd} \rangle \neq 0$. For a subset C of $X \times X$, we denote by $A(C)$ the matrix in $\text{Mat}_X(\mathbf{C})$ with (a, b) -entry equal to 1 if $(a, b) \in C$ and to 0 otherwise. Then we have the following:

Theorem 1. [5, Theorem 5(iii)] *Let C_1, \dots, C_p be the connected components of G . Then the algebra $\mathcal{N}(W^T)$ has a basis $\{A(C_i) \mid i = 1, \dots, p\}$.*

3. HADAMARD GRAPHS AND DIRECTED HADAMARD GRAPHS

In this section, we define the adjacency matrices of Hadamard graphs and directed Hadamard graphs, and give the association schemes determined by them. We refer the reader to [1, Theorem 1.8.1] for properties of Hadamard graphs, and to [2] for background materials in the theory of association schemes.

Let k be a positive integer, and let $H \in \text{Mat}_X(\mathbf{C})$ be a Hadamard matrix of order k . We denote by I_n the identity matrix of order n , and we omit n if $n = k$. Let $J \in \text{Mat}_X(\mathbf{C})$ be the all 1's matrix. We define

$$\begin{aligned} A_0 &= I_{4k}, \\ A_1 &= \begin{bmatrix} 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} J-I & J-I & 0 & 0 \\ J-I & J-I & 0 & 0 \\ 0 & 0 & J-I & J-I \\ 0 & 0 & J-I & J-I \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \end{bmatrix}, \\ A_4 &= \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}. \end{aligned}$$

The matrices $\{A_i\}_{i=0}^4$ are the distance matrices of the Hadamard graph associated to the Hadamard matrix H . Since the Hadamard graph is distance-regular (see [1, Theorem 1.8.1]),

$$(2) \quad \mathcal{A} = \text{span}\{A_0, A_1, A_2, A_3, A_4\}$$

is closed under the matrix multiplication. This algebra is called the Bose–Mesner algebra of the Hadamard graph.

The directed Hadamard graph associated with a Hadamard matrix H is the digraph with adjacency matrix

$$A'_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \end{bmatrix}.$$

The matrix A'_1 generates a Bose–Mesner algebra \mathcal{A}' with basis $A_0, A'_1, A_2, A'_3 = A_1'^T, A_4$ (see [6]). We shall index the rows and columns of the matrices A_1 and A'_1 by $X \times (\mathbf{Z}/2\mathbf{Z})^2$, in the order $X \times \{(0,0)\}$, $X \times \{(0,1)\}$, $X \times \{(1,0)\}$, $X \times \{(1,1)\}$. Then \mathcal{A} is the Bose–Mesner algebra of the

association scheme on $X \times (\mathbf{Z}/2\mathbf{Z})^2$ with relations

$$\begin{aligned} R_0 &= \{((a, \alpha), (a, \alpha)) \mid (a, \alpha) \in X \times (\mathbf{Z}/2\mathbf{Z})^2\}, \\ R_1 &= \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (0, 1), H(a, b) = (-1)^{\alpha_2 + \beta_2}\} \\ &\quad \cup \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (1, 0), H(b, a) = (-1)^{\alpha_2 + \beta_2}\}, \\ R_2 &= \{((a, \alpha), (b, \beta)) \mid \alpha_1 = \beta_1, a \neq b\}, \\ R_3 &= \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (0, 1), H(a, b) = (-1)^{\alpha_2 + \beta_2 + 1}\} \\ &\quad \cup \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (1, 0), H(b, a) = (-1)^{\alpha_2 + \beta_2 + 1}\}, \\ R_4 &= \{((a, \alpha), (a, \beta)) \mid \alpha_1 = \beta_1, \alpha_2 \neq \beta_2\}. \end{aligned}$$

Also, \mathcal{A}' is the Bose–Mesner algebra of an association scheme on $X \times (\mathbf{Z}/2\mathbf{Z})^2$ with relations $R_0, R'_1, R_2, R'_3, R_4$, where

$$\begin{aligned} R'_1 &= \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (0, 1), H(a, b) = (-1)^{\alpha_2 + \beta_2}\} \\ &\quad \cup \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (1, 0), H(b, a) = (-1)^{\alpha_2 + \beta_2 + 1}\}, \\ R'_3 &= \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (0, 1), H(a, b) = (-1)^{\alpha_2 + \beta_2 + 1}\} \\ &\quad \cup \{((a, \alpha), (b, \beta)) \mid (\alpha_1, \beta_1) = (1, 0), H(b, a) = (-1)^{\alpha_2 + \beta_2}\}. \end{aligned}$$

Let

$$\begin{aligned} Z_0 &= X \times \{0\} \times \mathbf{Z}/2\mathbf{Z}, \\ Z_1 &= X \times \{1\} \times \mathbf{Z}/2\mathbf{Z}, \\ Z &= Z_0 \cup Z_1, \\ R_i^0 &= R_i \cap (Z_0 \times Z), \\ R_i^1 &= R_i \cap (Z_1 \times Z). \end{aligned}$$

Let

$$\mathcal{R} = \{R_i^0 \mid i = 0, \dots, 4\} \cup \{R_i^1 \mid i = 0, \dots, 4\}.$$

Then \mathcal{R} is a coherent configuration in the sense of [3]. Let $R'_1 = R_1^0 \cup R_3^1$, $R'_3 = R_1^1 \cup R_3^0$. Then

$$R_i^\lambda R_j^\mu = \delta_{i+\lambda \bmod 2, \mu} \sum_{k \equiv i+j \pmod{2}} p_{ij}^k R_k^\lambda.$$

It follows that the permutation ρ of \mathcal{R} defined by

$$\rho(R_i^\delta) = \begin{cases} R_3^{1-\delta} & \text{if } i = 1, \\ R_1^{1-\delta} & \text{if } i = 3, \\ R_i^{1-\delta} & \text{otherwise} \end{cases}$$

is an algebraic automorphism of the coherent configuration \mathcal{R} in the sense of [8]. Since the relations $\mathcal{R}' = \{R_0, R'_1, R_2, R'_3, R_4\}$ are obtained

by fusing ρ -orbits, the fact that \mathcal{R}' forms an association scheme follows also from the general theory given in [8, Subsection 2.6].

4. SYMMETRIC AND NONSYMMETRIC HADAMARD MODELS

Throughout this section, we assume that k is an integer with $k \geq 4$. Let u be a complex number satisfying

$$(3) \quad k = (u^2 + u^{-2})^2$$

A Potts model $A \in \text{Mat}_X(\mathbf{C}^*)$ is defined as

$$(4) \quad A = u^3 I - u^{-1}(J - I).$$

Let $H \in \text{Mat}_X(\mathbf{C})$ be a Hadamard matrix of order k . In [9], [6], the following two spin models are given:

$$(5) \quad W = \begin{bmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega H^T & -\omega H^T & A & A \\ -\omega H^T & \omega H^T & A & A \end{bmatrix} \\ = u^3 A_0 + \omega A_1 - u^{-1} A_2 - \omega A_3 + u^3 A_4,$$

$$(6) \quad W' = \begin{bmatrix} A & A & \xi H & -\xi H \\ A & A & -\xi H & \xi H \\ -\xi H^T & \xi H^T & A & A \\ \xi H^T & -\xi H^T & A & A \end{bmatrix} \\ = u^3 A_0 + \xi A'_1 - u^{-1} A_2 - \xi A'_3 + u^3 A_4,$$

where ω is a 4-th root of unity, ξ is a primitive 8-th root of unity, respectively. We index the rows and columns of the matrices (5), (6) by $X \times (\mathbf{Z}/2\mathbf{Z})^2$ as in Section 3. The spin models (5) and (6) are called a Hadamard model and a nonsymmetric Hadamard model, respectively. From [5, Subsection 5.5] we have

$$(7) \quad \mathcal{N}(W) = \mathcal{A}.$$

In order to determine the Nomura algebra $\mathcal{N}(W')$, we consider the normalized version of the matrices (5), (6):

$$(8) \quad \tilde{W} = \begin{bmatrix} A & A & H & -H \\ A & A & -H & H \\ H^T & -H^T & A & A \\ -H^T & H^T & A & A \end{bmatrix},$$

$$(9) \quad \tilde{W}' = \begin{bmatrix} A & A & H & -H \\ A & A & -H & H \\ iH^T & -iH^T & A & A \\ -iH^T & iH^T & A & A \end{bmatrix},$$

where $i = -\xi^2$ is a primitive 4th root of unity. Then $\mathcal{N}(W) = \mathcal{N}(\tilde{W})$ and $\mathcal{N}(W') = \mathcal{N}(\tilde{W}')$, since

$$\tilde{W} = \begin{bmatrix} I & & 0 \\ & I & \\ & & \omega I \\ 0 & & & \omega I \end{bmatrix} W \begin{bmatrix} I & & 0 \\ & I & \\ & & \omega^{-1} I \\ 0 & & & \omega^{-1} I \end{bmatrix}$$

if $\omega^2 = 1$,

$$\tilde{W} = \begin{bmatrix} I & & 0 \\ & I & \\ & & \omega I \\ 0 & & & \omega I \end{bmatrix} W \begin{bmatrix} I & & 0 \\ & I & \\ & & \omega^{-1} I \\ 0 & & & \omega^{-1} I \end{bmatrix}$$

if $\omega^2 = -1$, and

$$\tilde{W}' = \begin{bmatrix} I & & 0 \\ & I & \\ & & \xi I \\ 0 & & & \xi I \end{bmatrix} W' \begin{bmatrix} I & & 0 \\ & I & \\ & & \xi^{-1} I \\ 0 & & & \xi^{-1} I \end{bmatrix}$$

(see [5, Propositions 2 and 3]).

Define column vectors $Y_{ab}^{\alpha\beta}$, $Y'_{ab}{}^{\alpha\beta}$ whose x -entries are given by

$$(10) \quad Y_{ab}^{\alpha\beta}(x) = \frac{\tilde{W}(x, (a, \alpha))}{\tilde{W}(x, (b, \beta))},$$

$$(11) \quad Y'_{ab}{}^{\alpha\beta}(x) = \frac{\tilde{W}'(x, (a, \alpha))}{\tilde{W}'(x, (b, \beta))}$$

for $(a, \alpha), (b, \beta) \in X \times (\mathbf{Z}/2\mathbf{Z})^2$, $x \in X \times (\mathbf{Z}/2\mathbf{Z})^2$.

Lemma 2. *Let G and G' be the graphs with the same vertex set $(X \times (\mathbf{Z}/2\mathbf{Z})^2)^2$, where two distinct vertices $((a, \alpha), (b, \beta)), ((a', \alpha'), (b', \beta'))$ are adjacent whenever*

$$\begin{aligned} \langle Y_{ab}^{\alpha\beta}, Y_{a'b'}^{\alpha'\beta'} \rangle &\neq 0, \\ \langle Y'^{\alpha\beta}_{ab}, Y'^{\alpha'\beta'}_{a'b'} \rangle &\neq 0, \end{aligned}$$

respectively. Let K_j ($j = 1, \dots, p$) (resp. K'_j ($j = 1, \dots, p'$)) be the connected components of G (resp. G'). Then $\mathcal{N}(W)$ (resp. $\mathcal{N}(W')$) is spanned by $\{A(K_j) \mid j = 1, \dots, p\}$ (resp. $\{A(K'_j) \mid j = 1, \dots, p'\}$).

Proof. By [5, Section 5.5], (7) holds regardless of the value of ω , so we have $\mathcal{N}(W) = \mathcal{N}(\tilde{W})$. Since $\tilde{W} = \tilde{W}^T$, the result for $\mathcal{N}(W)$ follows immediately from Theorem 1.

Since

$$\tilde{W}'^T = \begin{bmatrix} I_{2k} & 0 \\ 0 & -\xi^{-1}I_{2k} \end{bmatrix} W' \begin{bmatrix} I_{2k} & 0 \\ 0 & -\xi I_{2k} \end{bmatrix},$$

we have $\mathcal{N}(W') = \mathcal{N}(\tilde{W}'^T)$ by [5, Proposition 2]. Thus, the result for $\mathcal{N}(W')$ follows also from Theorem 1. \square

Let

$$D = \begin{bmatrix} I & & 0 \\ & I & \\ & & iI \\ 0 & & & iI \end{bmatrix} \in \text{Mat}_{X \times (\mathbf{Z}/2\mathbf{Z})^2}(\mathbf{C}).$$

Lemma 3. *For $(a, \alpha), (b, \beta) \in X \times (\mathbf{Z}/2\mathbf{Z})^2$,*

$$Y'^{\alpha\beta}_{ab} = \begin{cases} Y_{ab}^{\alpha\beta} & \text{if } \alpha_1 = \beta_1, \\ DY_{ab}^{\alpha\beta} & \text{if } (\alpha_1, \beta_1) = (0, 1), \\ D^{-1}Y_{ab}^{\alpha\beta} & \text{if } (\alpha_1, \beta_1) = (1, 0). \end{cases}$$

Proof. Immediate from the definitions (8), (9), (10) and (11). \square

Lemma 4. *Let τ denote the permutation of $(\mathbf{Z}/2\mathbf{Z})^2$ defined by*

$$\tau(\alpha_1, \alpha_2) = (\alpha_1, \alpha_1 + \alpha_2),$$

and let σ denote the permutation of $(X \times (\mathbf{Z}/2\mathbf{Z})^2)^2$ defined by

$$\sigma((a, \alpha), (b, \beta)) = ((a, \tau(\alpha)), (b, \beta)).$$

Then $\sigma(R_1) = R'_1$ and $\sigma(R_3) = R'_3$.

Proof. Immediate from the definitions of R_1, R_3, R'_1 and R'_3 . \square

Lemma 5. *For $(a, \alpha), (b, \beta) \in X \times (\mathbf{Z}/2\mathbf{Z})^2$,*

$$Y_{ab}^{\tau(\alpha)\beta} = (-D^2)^{\alpha_1} Y_{ab}^{\alpha\beta}.$$

Proof. Since $\tau(\alpha) = \alpha$ when $\alpha_1 = 0$, the result holds in this case. If $\alpha_1 = 1$, then $\tau(\alpha) = (1, 1 + \alpha_2)$. Since the (a, α) -column and $(a, \tau(\alpha))$ -column of \tilde{W} differ by the left multiplication by $-D^2$, the results holds in this case as well. \square

Lemma 6. *If $((a, \alpha), (b, \beta)), ((a', \alpha'), (b', \beta')) \in R_1 \cup R_3$, then*

$$\langle Y_{ab}^{\tau(\alpha)\beta}, Y_{a'b'}^{\tau(\alpha')\beta'} \rangle = (-1)^{\alpha_1 + \alpha'_1} \langle Y_{ab}^{\alpha\beta}, Y_{a'b'}^{\alpha'\beta'} \rangle.$$

Proof. Using Lemmas 3 and 5, we have

$$\begin{aligned} & \langle Y_{ab}^{\tau(\alpha)\beta}, Y_{a'b'}^{\tau(\alpha')\beta'} \rangle \\ &= \begin{cases} \langle DY_{ab}^{\tau(\alpha)\beta}, DY_{a'b'}^{\tau(\alpha')\beta'} \rangle & \text{if } (\tau(\alpha)_1, \beta_1) = (\tau(\alpha')_1, \beta'_1) = (0, 1), \\ \langle D^{-1}Y_{ab}^{\tau(\alpha)\beta}, D^{-1}Y_{a'b'}^{\tau(\alpha')\beta'} \rangle & \text{if } (\tau(\alpha)_1, \beta_1) = (\tau(\alpha')_1, \beta'_1) = (1, 0), \\ \langle DY_{ab}^{\tau(\alpha)\beta}, D^{-1}Y_{a'b'}^{\tau(\alpha')\beta'} \rangle & \text{if } (\tau(\alpha)_1, \beta_1) = (\beta'_1, \tau(\alpha')_1) = (0, 1), \\ \langle D^{-1}Y_{ab}^{\tau(\alpha)\beta}, DY_{a'b'}^{\tau(\alpha')\beta'} \rangle & \text{if } (\tau(\alpha)_1, \beta_1) = (\beta'_1, \tau(\alpha')_1) = (1, 0) \end{cases} \\ &= \begin{cases} \langle Y_{ab}^{\tau(\alpha)\beta}, Y_{a'b'}^{\tau(\alpha')\beta'} \rangle & \text{if } (\alpha_1, \beta_1) = (\alpha'_1, \beta'_1), \\ \langle D^2Y_{ab}^{\tau(\alpha)\beta}, Y_{a'b'}^{\tau(\alpha')\beta'} \rangle & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle (-D^2)^{\alpha_1}Y_{ab}^{\alpha\beta}, (-D^2)^{\alpha'_1}Y_{a'b'}^{\alpha'\beta'} \rangle & \text{if } \alpha_1 = \alpha'_1, \\ \langle D^2(-D^2)^{\alpha_1}Y_{ab}^{\alpha\beta}, (-D^2)^{\alpha'_1}Y_{a'b'}^{\alpha'\beta'} \rangle & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle Y_{ab}^{\alpha\beta}, Y_{a'b'}^{\alpha'\beta'} \rangle & \text{if } \alpha_1 = \alpha'_1, \\ -\langle Y_{ab}^{\alpha\beta}, Y_{a'b'}^{\alpha'\beta'} \rangle & \text{otherwise} \end{cases} \\ &= (-1)^{\alpha_1 + \alpha'_1} \langle Y_{ab}^{\alpha\beta}, Y_{a'b'}^{\alpha'\beta'} \rangle. \end{aligned}$$

\square

Theorem 7. *The Nomura algebra $\mathcal{N}(W')$ of the spin model W' coincides with the Bose–Mesner algebra \mathcal{A}' of the directed Hadamard graph.*

Proof. Since $u^4 = 1$ or $|u| > 1$, the coefficients $\{\xi, -u^{-1}, -\xi, u^3\}$ of W' in A'_1, A'_2, A'_3, A'_4 are pairwise distinct. Since $W' \in \mathcal{N}(W')$ by [5, Proposition 9], we obtain $A'_1, A'_2, A'_3, A'_4 \in \mathcal{N}(W')$. By Lemma 2, this implies that each of R'_1, R'_2, R'_3, R'_4 is a union of connected components of G' .

Since $\mathcal{N}(W) = \mathcal{A}$, Lemma 2 implies that R_0, R_1, \dots, R_4 are the connected components of G . Observe

$$R_0 \cup R_2 \cup R_4 = \{((a, \alpha), (b, \beta)) \in (X \times (\mathbf{Z}/2\mathbf{Z})^2)^2 \mid \alpha_1 = \beta_1\}.$$

By Lemma 3, we have

$$Y_{ab}^{\alpha\beta} = Y_{a'b'}^{\alpha'\beta'} \quad \text{for } ((a, \alpha), (b, \beta)) \in R_0 \cup R_2 \cup R_4.$$

This implies that two graphs G and G' have the same set of edges on $R_0 \cup R_2 \cup R_4$. Thus R_0 , R_2 and R_4 are connected.

By Lemmas 4 and 6, there is an isomorphism σ from the subgraph of G induced by $R_1 \cup R_3$, to the subgraph of G' induced by $R'_1 \cup R'_3$, satisfying $\sigma(R_1) = R'_1$ and $\sigma(R_3) = R'_3$. Since R_1 and R_3 are connected components of G , R'_1 and R'_3 are connected.

Therefore, we have shown that $R_0, R'_1, R_2, R'_3, R_4$ are the connected components of G' . The result now follows from Lemma 2. \square

Remark 8. The condition of Theorem 7 does not hold when $k = 1, 2$. A direct calculation shows that $\mathcal{N}(W) = \mathcal{N}(W')$ is the Bose–Mesner algebra of the group association scheme of $\mathbf{Z}/4\mathbf{Z}$ when $k = 1$, and that $\mathcal{N}(W) \cong \mathcal{N}(W')$ is the Bose–Mesner algebra of the group association scheme of $\mathbf{Z}/8\mathbf{Z}$ when $k = 2$.

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